

# NETWORK STRUCTURES DYNAMICS

## An ecological application

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- In this talk we present a topological description of the ecological transformations, an application of a multiagent ago-antagonist model in order to describe an ecosystem and a related stochastic model.
- An ecological system  $E$  is composed by different interacting communities as a set of species. Our idea is to consider the ecological niches as a basins where some species **go in** and others **go out**.
- *Network structures* have been recognized as one of a most suitable mathematical tools to model the interactions among the elementary components of *complex systems*.

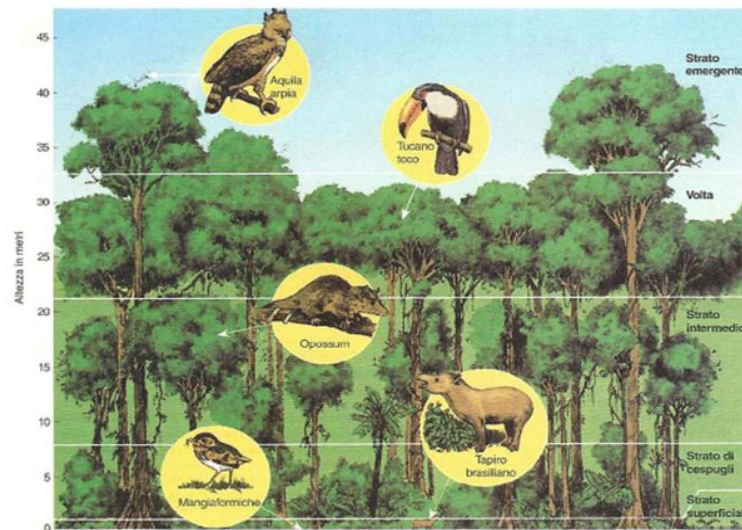
- Phenotypically an ecological system can be represented by a subset  $\mathbf{G}$  of  $N$  – dimensional Euclidean space, where  $N$  is the maximum possible number of phenotypical characters ( $\neq 0$ ) related to the individuals which belong to  $\mathbf{E}$ .  $\mathbf{E}$  is characterized to have a number of ecological niches  $\beta_j$  (where in a  $j$ -community several species live together) and possible paths from a niche to another.
- Utilizing in this description a network-graph (by means of  $\Phi$ -expressions) one has a so called *phenotypical fitness landscape on  $\mathbf{G}$* .

We denote by  $\alpha_j \subset \beta_j$  the *thick edge* of  $\beta_j$ .  $\alpha_j$  is called *periphery* of the niche  $\beta_j$  and  $\beta_j - \alpha_j$  is called *nucleus* of  $\beta_j$ . In general  $\beta_j - \alpha_j \neq \emptyset$

Niche as packing of species

#### Meccanismi di coesistenza

##### Nicchia ecologica: "impacchettamento" di specie



Stratificazione di piante specializzate e di nicchie animali a vari livelli di una foresta tropicale pluviale (da Miller 1997).

Queste nicchie specializzate permettono alle specie di evitare o minimizzare la competizione per le risorse con le altre e permettono la coabitazione di una grande varietà di specie.

La specializzazione della nicchia è regolata dall'adattamento delle piante a livelli diversi di luce disponibile negli strati della foresta e da centinaia di migliaia di anni di evoluzione in un clima abbastanza costante.

- If  $A_i^n \subset \mathbf{G}$  and  $B_u^n \subset \mathbf{G}$  (with  $n \leq N$ ) are two sets of the  $n$ -tuples, which belong to a subset of  $\mathbf{R}^+$  and which determine the values of **phenotypical characters** of the individuals respectively of a species  $i$  and of the species  $u$ . Of course these  $n$ -tuples depend on  $t$ .
- Let  $a_i^\wedge(t_s) \in \mathbf{F} \subset \mathbf{R}^+$  be a **phenotype-size** associated to a  $n$ -tuple of  $A_i^n \subset \mathbf{G}$ ; likewise  $b_u^\wedge$  for  $B_u^n \subset \mathbf{G}$ , we can consider the functions:

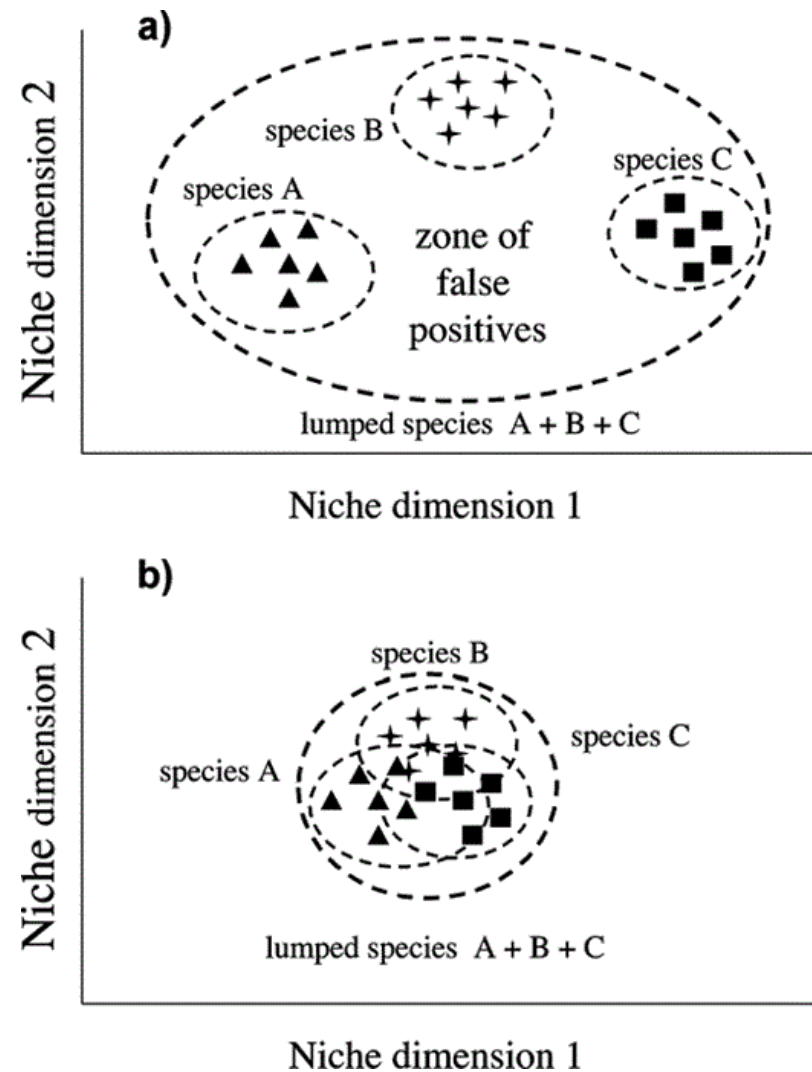
$$(1) \quad x_i(a_i^\wedge, \bar{t}_s) : \{a_i^\wedge, \bar{t}_s\} \subset \beta_j \rightarrow \{a_i'^\wedge, \bar{t}_r\} \subset \alpha_j \subset \beta_j$$

Likewise

$$y_u(b_u^\wedge, \bar{t}_s) : \{b_u^\wedge, \bar{t}_s\} \subset \alpha_j \rightarrow \{b_u'^\wedge, \bar{t}_r\} \subset \beta_j - \alpha_j \subset \beta_j$$

- The *phenotype transformations into the niche  $\beta_j$*  are determined by previous functions  $x_i$  and  $y_u$ . The niche  $\beta_j$  is constituted by the set phenotype-sizes of its individuals.
- The functions  $x_i$  and  $y_u$  establish for every considered phenotype size, a little alteration (during the time) of its value, which does not go far the specific range of compatibility of the species  $i$  or  $u$ . I.e., the passage (through  $y_u$ ) from  $\alpha_j$  to  $\beta_j - \alpha_j$  of a phenotype is represented by an alteration of the value which makes the phenotype (individuals) compatible with  $\beta_j - \alpha_j$ . Because every individual of a species has an its own phenotype value, these transformations-functions  $x_i$  and  $y_u$  regard at the time  $t_s$  the set of individuals (Domain of  $x_i$  or  $y_u$ ) of the considered species which stay or in periphery or in the nucleus of the niche. But it is possible that some individuals of the same species have not a phenotype alteration, while other individuals have an alteration. In this case it is necessary to distinguish on the functions  $x_i$  or  $y_u$ .
- It is easy to establish for  $x_i$  and  $y_u$  the notion of *velocity of transformation*

- In the case a) of the nearby schema exemple, the species A, B, C are lumped in the periphery  $\alpha_j \subset \beta_j$ . In this case  $\beta_j - \alpha_j = \emptyset$ , but  $\alpha_j \neq \emptyset$ . This ecological situation means that the functions  $x_i$  prevail and the niche bicommes **very fragile**.
- In the case b) the species A, B, C are lumped in the nucleus  $\beta_j - \alpha_j \neq \emptyset$  and  $\alpha_j = \emptyset$ . In this ecological situation the functions  $y_u$  prevail and the niche bicommes **very robust**.



## • $\Phi$ - EXPRESSIONS

A network structure can be represented by means of a compact symbolism. This symbolism is established by expressions which we will call  $\Phi$  - expressions. This symbolism is based on the graph theory (the literature is very wide, i.e. Oliver Cogis & Claudine Robert, James R. Peterson (Petri net theory), Mark Buchanam (Nexus theory), etc.). According to our interpretation, the vertices  $\cdot$  of a graph (network structure) represent the **Domain** or the **Codomain** of the functions  $x_i$  or  $y_u$ . The arcs  $\rightarrow$  of the graph represent the previous functions. So e.g.  $\cdot_s \rightarrow \cdot_r$  denotes :

$$s = \text{Dom}x_i \rightarrow r = \text{Codom}x_i$$

Likewise for  $y_u$ , by  $\cdot_r \leftarrow \cdot_s$



Now we gave some exemples of this symbolism.

$$(2) \quad \langle {}^m\Phi_s^n \rangle x_i, y_j$$

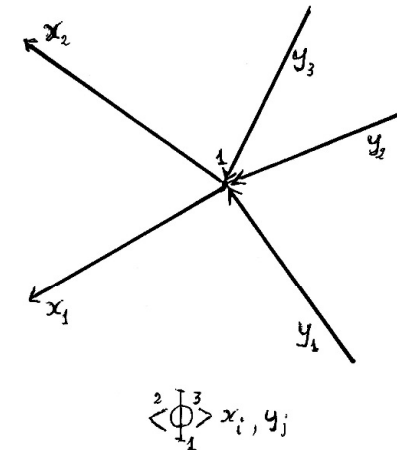
where  $s$  denotes the «name» of a considered vertex, which represents a *basin*

$m$  is the number of  $x_i$ , that is the number of the arcs which go out from  $s$

$n$  is the number of arcs  $y_u$  which go in  $s$ .

Of course  $0 < i < m$  and  $0 < u < n$ .

See figure when  $s = 1$ ,  $m = 2$  and  $n = 3$



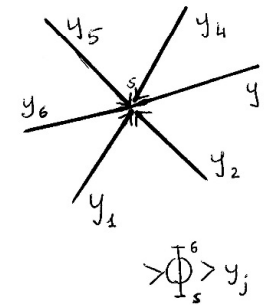
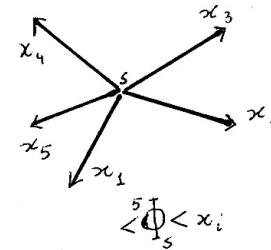
- We have also two other kinds of *basin*:

$$<^m\Phi_s < x_i$$

and the case

$$>\Phi_s^n > y_j$$

See near here the related figures



We have also two kinds of *paths*:

*x-path*

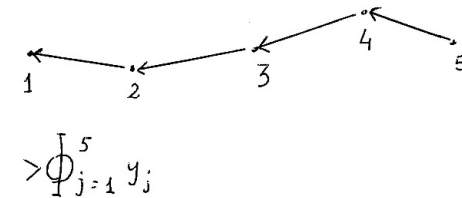
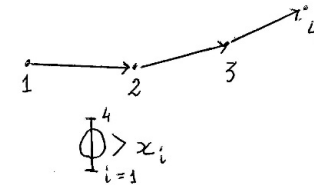
$$(3) \quad \Phi_{i=1}^n > x_i$$

where  $i$ , in this case, denotes the «names» of  $n + 1$  vertices connected by arcs  $x_i$  in sequence with the same direction

*y-path*

$$> \Phi_{j=1}^m y_j$$

where  $j$ , in this case, denotes the «names» of  $m + 1$  vertices connected by arcs  $y_j$  in sequence with the same direction, but opposite of the previous



- Connexions among basins and paths are possible. As symbol of connexion **in** or **from** the vertex  $i$  we utilize  $\oplus_i$ , so, for instance:

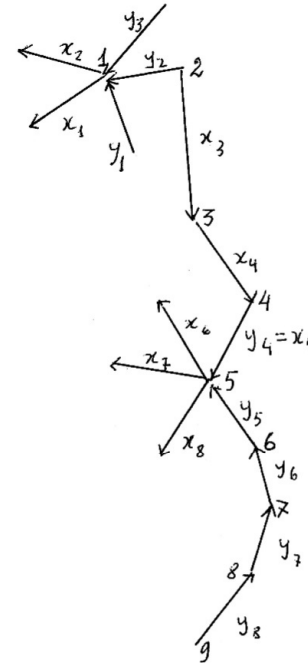
$$(<^2\Phi^3_1>x_i, y_j) \oplus_2 (\Phi^4_{i=2}>x_i) \oplus_4 \\ \oplus_4 (<^3\Phi^2_5>x_i, y_j) \oplus_6 (>\Phi^9_{j=6}y_j)$$

with  $y_4 = x_5$

See near here figure

NB: We have the following syntax rule: If  $\Phi_i$  and  $\Phi_j$  are two  $\Phi$ -expressions then

$\Phi_i \oplus_k \Phi_j$  is a  $\Phi$ -expression



- When one studies a network structure, at first one must state the  $\Phi$ -expression which is considered in order to describe the situation among the previous functions.

- DETERMINISTIC DYNAMICS

- In general, even if we consider i.e. a basin  $\langle^m \Phi^n_s \rangle x_i, y_j$ , it is possible establish the following dynamics. We will consider a general case of degree  $n$  of paroxysm according the Cherruault model.

$$(4) \quad \left\{ \begin{array}{l} \dot{x}_1 = a_{11} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right) + a_{12} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^2 + \dots + a_{1n} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^n \\ \dots \\ \dot{x}_n = a_{n1} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right) + a_{n2} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^2 + \dots + a_{nn} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^n \\ \dot{y}_1 = b_{11} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right) + b_{12} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^2 + \dots + b_{1n} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^n \\ \dots \\ \dot{y}_n = b_{n1} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right) + b_{n2} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^2 + \dots + b_{nn} \left( \sum_{i=1}^n x_i - \sum_{i=1}^n y_i \right)^n \end{array} \right.$$

- If in the previous system we subtract member to member, we have:

$$\sum_{i=1}^n (\dot{x}_i - \dot{y}_i) = \sum_{i=1}^n (a_{i1} - b_{i1}) \sum_{i=1}^n (x_i - y_i) + \sum_{i=1}^n (a_{i2} - b_{i2}) \sum_{i=1}^n (x_i - y_i)^2 + \dots + \sum_{i=1}^n (a_{in} - b_{in}) \sum_{i=1}^n (x_i - y_i)^n$$

We will call

$$(5) \quad z = \sum_{i=1}^{n_1} x_i - \sum_{j=1}^{n_2} y_j \quad \text{out-in balance}$$

If we put

$$\sum_{i=1}^n (\dot{x}_i - \dot{y}_i) = \dot{z}$$

$$\sum_{i=1}^n (a_{ij} - b_{ij}) = c_j$$

$$\sum_{i=1}^n (x_i - y_i) = z$$

- We obtain:

$$(6) \quad \dot{z} = \sum_{i=1}^n c_i z^i$$

- Now one can consider  $z$  and  $t$  as two independent variables and one can consider a two independent variables function  $W(z, t)$ , for instance so

$$(7) \quad W(z, t) = - \sum_{i=2}^n \frac{c_{i-1} z^i}{i}$$

which we call *out-in potential*. Hence the previous equation (5) bicommes

$$(8) \quad \dot{z} = - \frac{\partial W(z, t)}{\partial z}$$



- The condition in order to have a basin (i.e. the center of an ecological niche) is to have local **minima** of the potential  $W(z, t)$ . We have also the following classification:

$\langle^m \Phi_s^n \rangle_{x_i, y_j} : \text{semiattractive or semirepulsive basin or niche}$

$\langle^m \Phi_s \rangle_{x_i} : \text{repulsive basin or niche}$

$\rangle \Phi_s^n \rangle_{y_j} : \text{attractive basin or niche}$

We could study this mathematical representation in the context of a **functional space**

- In conclusion, we have a basin when we have a **minimum**  $z = z_s$  for the out-in potential  $W(z, t)$ .
- But the value of  $W(z, t)$  in  $z = z_s$  depend on the mathematical form of the expression of the same  $W(z, t)$ . A discussion about this value is **linked** also the ecological situation which  $z$  expresses.

- For insance, when

$$z_s = \sum_{i=1}^{n_1} x_i \quad \text{that is} \quad \sum_{j=1}^{n_2} y_j = 0$$

the input-functions annul its effects, hence the niche tends to have above all species in the periphery  $\alpha_j \subset \beta_j$ . The  $j$ -community is **more fragile**.

- when

$$z_s = \sum_{i=1}^{n_1} y_i \quad \text{that is} \quad \sum_{j=1}^{n_2} x_j = 0$$

the out-functions annul its effects, so the niche tends to  $\beta_j - \alpha_j$  and the  $j$ -community become **more robust**.

- Moreover, when

$$\sum_{i=1}^{n_1} x_i = \sum_{j=1}^{n_2} y_j \Leftrightarrow z = 0$$

the niche has IN and OUT equivalence. We have a kind of equilibrium.

We state that if the phenotype-size of the individuals of a species A is greater than that of another species B then the species A is biologically more robust than B. So, when:

$$\sum_{i=1}^{n_1} x_i > \sum_{j=1}^{n_2} y_j \Leftrightarrow z > 0$$

both  $n_1 > n_2$  and  $n_1 < n_2$  the out- functions prevail and the niche is biologically **more fragile**. While, when

$$\sum_{i=1}^{n_1} x_i < \sum_{j=1}^{n_2} y_j \Leftrightarrow z < 0$$

the niche becomes **more robust** (the in-functions prevail).

- While for a *x-path* the *transformations* can be expressed respectively, for the case  $[\Phi^n_{i=1} > x_i]$ , by

$$\sum_{i=1}^n \dot{x}_i = \sum_{i=1}^n a_{i1} \sum_{i=1}^n x_i + \sum_{i=1}^n a_{i2} \sum_{i=1}^n x_i^2 + \dots + \sum_{i=1}^n a_{in} \sum_{i=1}^n x_i^n$$

and if one puts:

$$\sum_{i=1}^n \dot{x}_i = \dot{x} ; \sum_{i=1}^n a_{ij} = c_j ; \sum_{i=1}^n x_i = x$$

we have:

$$\dot{x} = \sum_{i=1}^n c_i x^i$$

And likewise for the case *y-path*  $[> \Phi^m_{j=1} y_j]$ :

$$\dot{y} = \sum_{i=1}^n c_i y^i$$

- Hence the **total transformations** of a considered network discrete structure can be described by a number  $r$  (which depends on the number  $r$  of **basins**) of basin-equations, a number  $s$  (which depends on the number  $s$  of ***x-paths***) of  $x$ -paths equations and a number  $u$  (which depends on the number  $u$  of ***y-paths***) of  $y$ -paths equations.
- In general an ecosystem is constitute by a number of niches linked through  $x$ -paths and  $y$ -paths. **These represent trajectories of ecological transformations.** It is interesting to study also when **isolated niches** exist.

- **STOCHASTIC DYNAMICS**

But the dynamics of our ecological systems is labeled by the behaviour of  $z$  also with the addition of a stochastic dynamics component, so (equation of Smoluchowski):

$$(9) \quad \dot{z} = -\frac{\partial W(z, t)}{\partial z} + \sqrt{2T} \xi(t)$$

where  $T$  is a *temperature* of the ecological system that measures the stability of the different species and  $\xi(t)$  is a white noise.

Without loss of generality we set  $W(z, t) \leq 0$  in the region of a basin, that is in the neighbourhood of  $z_s$ . So that the quantity

$$(10) \quad W_s = -W(z_s, t)/T$$

is the *deepness* of the potential well (of  $z_s$ ) using  $T$  as potential unit.

- Without the noise effect (i.e.  $T = 0$ ) we stay in the situation of (8) where any individual of the  $j$ -community is attracted by the critical point  $z_s$ , that is the individual stay in the nucleus  $\beta_j - \alpha_j$  or in periphery  $\alpha_j$ , but it does not go out from  $\beta_j$ .
- On the contrary, for  $T > 0$  each trajectory has the possibility to **jump** between potential wells, that is niches, modelling an interactions between the corresponding communities. In this case the topological description of the trajectory from a niche to another is represented by  $\Phi^n_{i=1} > x_i$  and /or  $> \Phi^m_{j=1} y_j$ .
- We define *success of the  $j$ -community*  $n^*_j$  the number of individuals of the different species that populate the corresponding niche.
- **Remark 1:** According to our interpretation of the model, the success of the  $j$ -community is directly proportional to the probability of finding a representative individual (standard representative individual) in the neighborhood of  $z_s$ .

- **Remark 2** : One can associate a stationary distribution probability  $P_{\text{st}}(z)$  to the stochastic dynamics (9) according to:

$$(11) \quad P_{\text{st}}(z) = A \exp (- W(z, t)/T)$$

- **Remark 3**: According to (11) in the stationary state the success  $n^*_j$  of the  $j$ -community is given by the deepness of the potential well (Maxwell-Boltzmann distribution)

$$(12) \quad n^*_j \propto \exp (W_j)$$

and the relation

$$(13) \quad \sum_{j=1, \dots, N} n^*_j = M_T$$

give the total number of the individuals in the ecosystem.



- **Remark 4:** One can prove that the *escape rate* from the  $W_j$ -potential well is proportional to (Arrhenius' law)

$$(14) \quad P_{\text{esc}, j}(z) \propto \exp(-W_j)$$

- **Remark 5:** A possible *definition* of the *interaction rate*  $\pi_{ij}$  of the community  $j$  with the community  $i$  depends on the related potential wells, so:

$$(15) \quad \pi_{ij} = \frac{1}{M} \exp(W_i - W_j) = \frac{1}{M} \frac{n_i^*}{n_j^*}$$

where  $M$  is the number of different communities.

- The Lotka-Volterra equations describe, in an effective way, the average dynamics of the community success  $n_i(t)$ . We impose the existence of stationary equilibrium  $n_i^*$  and we introduce a birth rate  $g$  (which represents the reproduction mechanism). The condition  $g > 0$  implies the stability for the stationary solution  $n_i^*$  and  $g$  is directly proportional to the exponential of the fitness. Because we consider  $M$  communities, the Lotka-Volterra equation becomes a system of equations. Hence the matrix

$$(\pi_{ij} - \delta_{ij}) = \begin{bmatrix} \pi_{11} & \dots & \pi_{1M} \\ \dots & \dots & \dots \\ \pi_{1M} & \dots & \pi_{MM} \end{bmatrix} - \begin{bmatrix} 1 & \dots & 0 \\ 0 & \dots & 0 \\ 0 & \dots & 1 \end{bmatrix}$$

with  $i = 1, \dots, M$ , represents a **cooperative interaction among the communities**

Therefore we write the **Lotka-Volterra dynamics** of the communities as

$$(16) \quad \begin{aligned} \dot{n}_i &= n_i \left[ g(n_i^* - n_i) + \sum_{j=1}^M (\pi_{ij} - \delta_{ij}) n_j \right] \\ i &= 1, \dots, M \end{aligned}$$

We are interested in the dynamics near the stationary state  $n_i \sim n_i^*$  in this case

$$n_i^* = \sum_{j=1}^M \pi_{ij} n_j^* \quad \text{with } i = 1, \dots, M$$

i.e.  $n^*$  is the eigenvector of the matrix  $\pi_{ij}$  with eigenvalue 1.

Now we denote by  $m_i$  the numerousness of a  $i$ - community or of  $i$ -species. And we denote by  $E_i^\pm$  the Van Kampen operators which increases by an unitary quantity the numerousness  $m_i$  , so i.e.:

$$E_i^\pm P(m) = P(m_1, \dots, m_i \pm 1, \dots, m_N)$$

and we can introduce the Master Equation so

$$\dot{P}(m, t) = \sum_{i=1}^N n_i^* (E_i^+ - 1) \left[ (g + 1) m_i P(m, t) - E_i^- \left( g + \frac{1}{N} \sum_{j=1}^N m_j \right) P(m, t) \right]$$

We have an explicit solution

$$P(m_1, \dots, m_N) = \prod_{i=1}^N (n_i^*)^{m_i} \prod_{k_i=1}^{m_i} \frac{Ng + \sum_{j=1}^N k_j - 1}{N(g+1)k_i} P_s(0)$$

That could be used to study the existence of different communities in a ecological system from empirical data.

- We have taken in account the following works:

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Chase JM, Leibold MA, *Ecological Niches. Linking Classical and Contemporary Approaches*, Chicago: University of Chicago Press, 2003 , 212 pp.

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Van Kampen NG, *Stochastic Process in Physics and Chemistry*, Amsterdam: North-Holland, 2001

Cogis O, Robert C, *Théorie des Graphes*, Paris: Vuibert, 2003