Pseudospectral methods for the stability of periodic solutions of delay equations

Davide Liessi
joint work with Dimitri Breda

CDLab – Computational Dynamics Laboratory
Department of Mathematics, Computer Science and Physics
University of Udine, Italy

9th workshop DSABNS 2018

Session on delayed equations

University of Torino, 7 February 2018

delay equations

*“A delay equation is a rule
for extending a function of time towards the future
on the basis of the (assumed to be) known past.”*

delay equations

$$0 < \tau < \infty \text{ delay} \qquad X = L^1([-\tau, 0], \mathbb{R}^{d_X})$$
$$d_X, d_Y \geqslant 0 \text{ integers} \quad Y = C([-\tau, 0], \mathbb{R}^{d_Y})$$
$$y_t(\theta) \coloneqq y(t + \theta), \quad \theta \in [-\tau, 0]$$

DDE (RFDE)

$$y'(t) = G(y_t), \quad G \colon Y \to \mathbb{R}^{d_Y}$$

RE (VFE)

$$x(t) = F(x_t), \quad F \colon X \to \mathbb{R}^{d_X}$$

coupled RE/DDE

$$\begin{cases} x(t) = F(x_t, y_t), & F \colon X \times Y \to \mathbb{R}^{d_X} \\ y'(t) = G(x_t, y_t), & G \colon X \times Y \to \mathbb{R}^{d_Y} \end{cases}$$

delay equations

motivation: the *Daphnia* model

$$\begin{cases} b(t) = \int\limits_{a_A(S_t)}^{h} \beta(X(a,S_t),S(t))\mathcal{F}(a,S_t)b_t(-a)\,da & \text{(birth rate)} \\ S'(t) = f(S(t)) - \int\limits_0^h \gamma(X(a,S_t),S(t))\mathcal{F}(a,S_t)b_t(-a)\,da & \text{(resource concentration)} \end{cases}$$

where

- $X(a, S_t)$ is the size of individual experienced resource history S_t
- $\mathcal{F}(a, S_t)$ is the survival probabili who has experienced resource hi
- $\beta(x, S)$ is the per capita fertility
- $\gamma(x, S)$ is the per capita consum
- $f(S)$ describes consumer-free res

[de Roos,

[Diekmann, Gyllenberg, M

R. Vermiglio nonlinear delay equations

Example from ecology

an age-structured population

Renewal equation for population birth rate

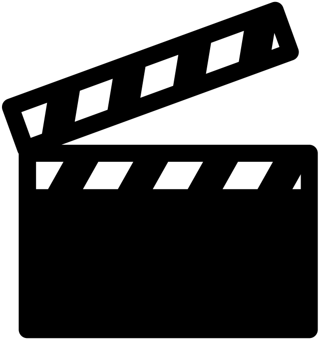
$$b(t) = \int_{a_{\mathrm{repr}}}^{a_{\max}} \overbrace{\beta(a)}^{\text{fertility}} \overbrace{\mathcal{F}(a)\, b(t-a)}^{\text{ind of age } a}\, da$$

often coupled with a delay-differential equation for the environmental variable (substrate, prey,...)

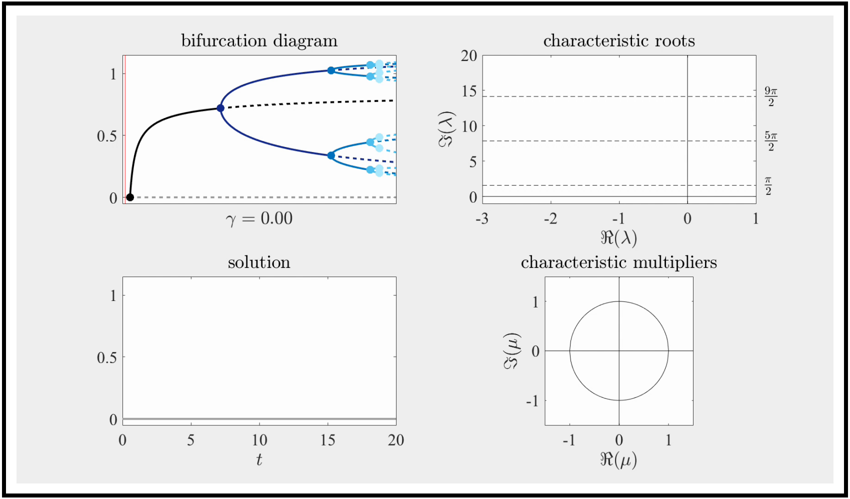
$$\frac{dS}{dt}(t) = \underbrace{f(S(t))}_{\text{consumer-free}} - \int_0^{a_{\max}} \overbrace{\gamma(a)}^{\text{consumption}} \overbrace{\mathcal{F}(a)\, b(t-a)}^{\text{ind of age } a}\, da$$

Francesca Scarabel (Helsinki) Numerical bifurcation analysis of infinite-delay equations in biology 3 / 23

a short animated movie [1]



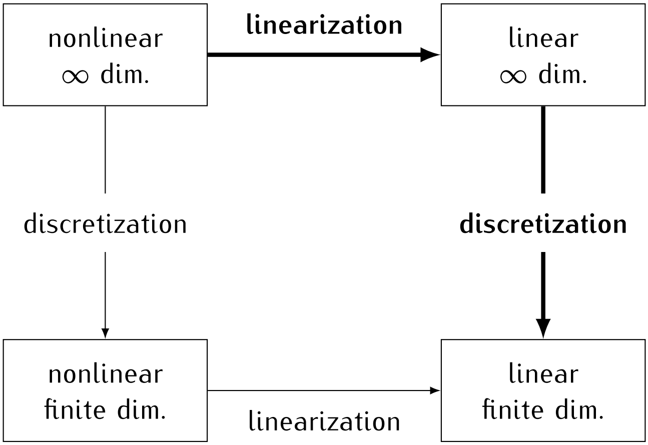
[1] Breda, Diekmann, Liessi, and Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ. 65 (2016), pp. 1–24, DOI:10.14232/ejqtde.2016.1.65.


bifurcation diagram
1
0.5
0
$\gamma = 0.00$
characteristic roots
20
15
10
5
0
$\Im(\lambda)$
$\frac{9\pi}{2}$
$\frac{5\pi}{2}$
$\frac{\pi}{2}$
-3
-2
-1
0
1
$\Re(\lambda)$
solution
1
0.5
0
0
5
10
15
20
t
characteristic multipliers
1
0
-1
$\Im(\mu)$
-1
0
1
$\Re(\mu)$

two complementary approaches

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\boxed{\begin{array}{c}\text{nonlinear}\\ \infty\ \text{dim.}\end{array}} \arrow[r, "\text{linearization}"] \arrow[d, "\text{discretization}" description] & \boxed{\begin{array}{c}\text{linear}\\ \infty\ \text{dim.}\end{array}} \arrow[d, "\text{discretization}" description] \\
\boxed{\begin{array}{c}\text{nonlinear}\\ \text{finite dim.}\end{array}} \arrow[r, "\text{linearization}"'] & \boxed{\begin{array}{c}\text{linear}\\ \text{finite dim.}\end{array}}
\end{tikzcd}
$$

two complementary approaches



- $(\bar{x}, \bar{y})$ periodic solution

$$L_{XX}(t) \coloneqq \frac{\partial F}{\partial x}(\bar{x}_t, \bar{y}_t) \qquad L_{XY}(t) \coloneqq \frac{\partial F}{\partial y}(\bar{x}_t, \bar{y}_t)$$
$$L_{YX}(t) \coloneqq \frac{\partial G}{\partial x}(\bar{x}_t, \bar{y}_t) \qquad L_{YY}(t) \coloneqq \frac{\partial G}{\partial y}(\bar{x}_t, \bar{y}_t)$$

- linearized IVP

$$\begin{cases} x(t) = L_{XX}(t)x_t + L_{XY}(t)y_t, & t > s \\ y'(t) = L_{YX}(t)x_t + L_{YY}(t)y_t, & t \geqslant s \\ x_s = \varphi, \quad y_s = \psi \end{cases}$$

evolution operators

- linearized IVP

$$\begin{cases} x(t) = \int_{-\tau}^{0} C_{XX}(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t, & t > s \\ y'(t) = \int_{-\tau}^{0} C_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t, & t \geqslant s \\ x_s = \varphi, \quad y_s = \psi \end{cases}$$

- family of evolution operators

(x, y) solution of linearized IVP, $t \geqslant s$

$$\begin{aligned} T(t,s)\colon X \times Y &\to X \times Y \\ \underset{\substack{\| \\ (x_s,\, y_s)}}{(\varphi,\psi)} &\mapsto (x_t, y_t) \end{aligned}$$

Floquet theory

- $\mathcal{M} \coloneqq T(\Omega, 0)$ monodromy operator, Ω period of $(\bar{x}, \bar{y})$

Floquet theory for DDE [3, 4] $\begin{cases} \text{if } 1 \in \sigma(\mathcal{M}) \text{ is simple} \\ \text{and } |\mu| < 1 \text{ for all } \mu \in \sigma(\mathcal{M}) \setminus \{1\}, \\ \Rightarrow \text{solution locally asymptotically stable} \end{cases}$

Floquet theory for RE/coupled? $\begin{cases} \text{work in progress based on [3, 5]} \\ \text{with O. Diekmann and S. M. Verduyn Lunel} \end{cases}$

[3] Diekmann, Van Gils, Verduyn Lunel, and Walther, *Delay Equations. Functional-, Complex-, and Nonlinear Analysis*, Appl. Math. Sci. 110, Springer, New York, 1995, DOI:10.1007/978-1-4612-4206-2.

[4] Hale and Verduyn Lunel, *Introduction to Functional Differential Equations*, 2nd ed., Appl. Math. Sci. 99, Springer, New York, 1993, DOI:10.1007/978-1-4612-4342-7.

[5] Diekmann, Getto, and Gyllenberg, *Stability and bifurcation analysis of Volterra functional equations in the light of suns and stars*, SIAM J. Math. Anal. 39 (2008), pp. 1023–1069, DOI:10.1137/060659211.

pseudospectral methods

- polynomial interpolation

$$R_M\psi = (\psi(\theta_0), \dots, \psi(\theta_M))$$

$$P_M\Psi = \sum_{j=0}^{M} l_j\Psi_j, \quad l_j(\theta) = \prod_{k\neq j} \frac{\theta - \theta_k}{\theta_j - \theta_k}$$

$$R_M P_M = I_{M+1}$$

$P_M R_M$ Lagrange interpolation operator

- pseudospectral methods

"Do to $P_M R_M \psi$ as you would do to ψ."

- spectral accuracy
  with "good" nodes (e.g., Chebyshev zeros or extrema)
  smooth functions: error $= O(M^{-k})$ for every k
  analytic functions: error $= O(c^M)$ for $0 < c < 1$

reconstructing the solution

$$T := T(s+h, s)$$

$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$

$$Y^+ := C([0,h], \mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

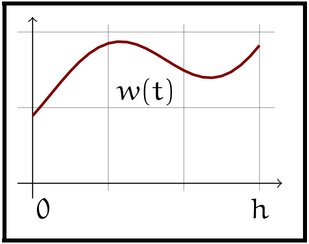
$$\begin{cases} x(t) = \int_{-\tau}^{0} C_{XX}(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t \\ y'(t) = \int_{-\tau}^{0} C_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t \end{cases}$$

$$V\colon (X \times Y) \times (X^+ \times Y^+) \to (X^\pm \times Y^\pm)$$

reconstructing the solution

$$T := T(s+h, s)$$

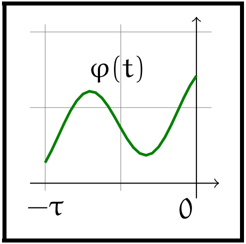
$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$



$$\ldots \mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

$$\ldots_X(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t$$

$$\ldots_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t$$



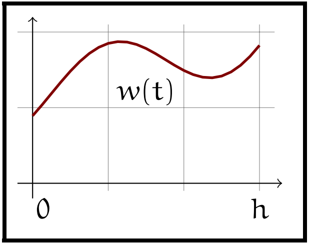
$$\ldots \times (X^+ \times Y^+) \to (X^\pm \times Y^\pm)$$

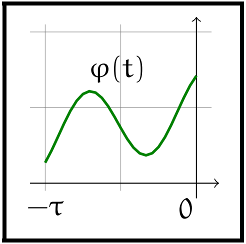
reconstructing the solution

$$T := T(s+h, s)$$

$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$

$$\mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

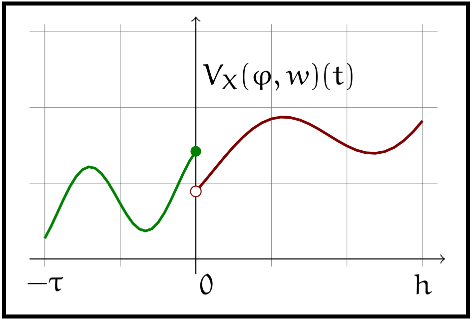




$x($

X

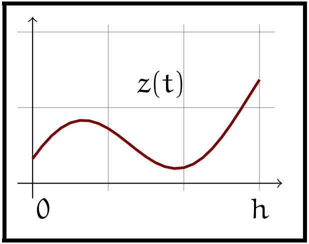
$\times\ (X$



reconstructing the solution

$$T := T(s+h, s)$$

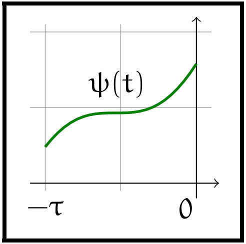
$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$



$${}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

$${}_X(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t$$

$${}_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t$$



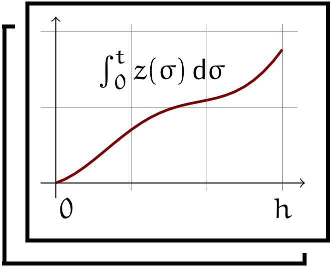
$$\times (X^+ \times Y^+) \to (X^\pm \times Y^\pm)$$

reconstructing the solution

$$T := T(s+h, s)$$

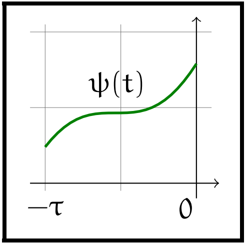
$$X) \quad X^{\pm} := L^1([-\tau, h], \mathbb{R}^{d_X})$$

$$) \quad Y^{\pm} := C([-\tau, h], \mathbb{R}^{d_Y})$$



$$t, \theta) x_t(\theta)\, d\theta + L_{XY}(t) y_t$$

$$(t, \theta) x_t(\theta)\, d\theta + L_{YY}(t) y_t$$



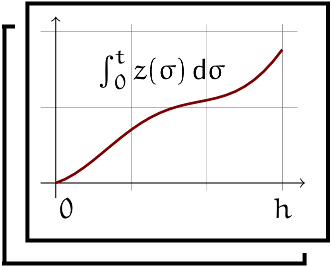
$$\times (X^{+} \times Y^{+}) \to (X^{\pm} \times Y^{\pm})$$

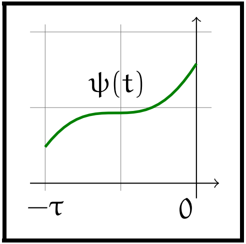
reconstructing the solution

$$T := T(s+h, s)$$

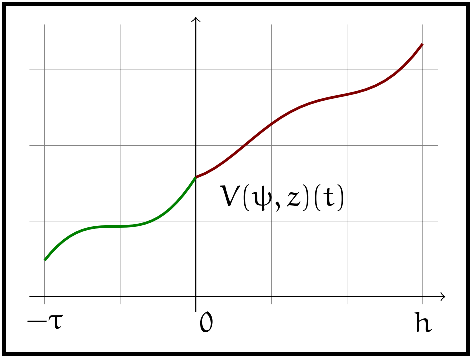
$x\,)$ $\quad X^{\pm} := L^1([-\tau, h], \mathbb{R}^{d_X})$

$)$ $\quad Y^{\pm} := C([-\tau, h], \mathbb{R}^{d_Y})$





$\times\ (X$



reconstructing the solution

$$T := T(s+h, s)$$

$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$

$$Y^+ := C([0,h], \mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

$$\begin{cases} x(t) = \int_{-\tau}^{0} C_{XX}(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t \\ y'(t) = \int_{-\tau}^{0} C_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t \end{cases}$$

$$V\colon (X \times Y) \times (X^+ \times Y^+) \to (X^\pm \times Y^\pm)$$

$$V((\varphi,\psi),(w,z))(t) = \begin{cases} \left(w(t), \psi(0) + \int_0^t z(\sigma)\,d\sigma\right) & t \in (0,h] \\ (\varphi(t), \psi(t)) & t \in [-\tau, 0] \end{cases}$$

reconstructing the solution

$$T := T(s+h, s)$$
$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$
$$Y^+ := C([0,h], \mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

$$\begin{cases} x(t) = \displaystyle\int_{-\tau}^{0} C_{XX}(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t \\ y'(t) = \displaystyle\int_{-\tau}^{0} C_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t \end{cases}$$

$$\mathcal{F}_s \colon X^\pm \times Y^\pm \to X^+ \times Y^+$$

reconstructing the solution

$$T := T(s+h, s)$$

$$X^+ := L^1([0,h], \mathbb{R}^{d_X}) \quad X^\pm := L^1([-\tau,h], \mathbb{R}^{d_X})$$

$$Y^+ := C([0,h], \mathbb{R}^{d_Y}) \quad Y^\pm := C([-\tau,h], \mathbb{R}^{d_Y})$$

$$\begin{cases} x(t) = \int_{-\tau}^{0} C_{XX}(t,\theta)x_t(\theta)\,d\theta + L_{XY}(t)y_t \\ y'(t) = \int_{-\tau}^{0} C_{YX}(t,\theta)x_t(\theta)\,d\theta + L_{YY}(t)y_t \end{cases}$$

$$\mathcal{F}_s \colon X^\pm \times Y^\pm \to X^+ \times Y^+$$

$$\mathcal{F}_s(u,v)(t) = \Big(\int_{-\tau}^{0} C_{XX}(s+t,\theta)u_t(\theta)\,d\theta + L_{XY}(s+t)v_t,$$

$t \in [0,h]$

$$\int_{-\tau}^{0} C_{YX}(s+t,\theta)u_t(\theta)\,d\theta + L_{YY}(s+t)v_t \Big)$$

solution operator approach

- reformulation of $T: X \times Y \to X \times Y$

$$T(\varphi, \psi) = V((\varphi, \psi), (w, z))_h$$
$$(w, z) = \mathcal{F}_s V((\varphi, \psi), (w, z))$$

- $\theta_0 > \theta_1 > \cdots > \theta_M$ Chebyshev extrema in $[-\tau, 0]$
  $X_M \coloneqq \mathbb{R}^{d_X(M+1)}$, $Y_M \coloneqq \mathbb{R}^{d_Y(M+1)}$, R_M, P_M

- $t_1 < \cdots < t_N$ Chebyshev zeros in $[0, h]$
  $X_N^+ \coloneqq \mathbb{R}^{d_X N}$, $Y_N^+ \coloneqq \mathbb{R}^{d_Y N}$, R_N^+, P_N^+

- discretization of T as $T_{M,N}: X_M \times Y_M \to X_M \times Y_M$

$$T_{M,N}(\Phi, \Psi) = R_M V(P_M(\Phi, \Psi), P_N^+(W, Z))_h$$
$$(W, Z) = R_N^+ \mathcal{F}_s V(P_M(\Phi, \Psi), P_N^+(W, Z))$$

convergence proof

- follows the lines of the DDE case [6, 7]
  - $C \rightsquigarrow L^1$
  - different definition of V
  - different regularization properties of $\mathcal{F}_s$ and V
- complete for RE [paper accepted on SIAM J. Numer. Anal.]
- complete for coupled RE/DDE [PhD thesis]

[6] Breda, Maset, and Vermiglio, *Approximation of eigenvalues of evolution operators for linear retarded functional differential equations*, SIAM J. Numer. Anal. 50 (2012), pp. 1456–1483, DOI:10.1137/100815505.

[7] Breda, Maset, and Vermiglio, *Stability of linear delay differential equations. A numerical approach with MATLAB*, Springer Briefs Control, Autom. and Robot., Springer, New York, 2015, DOI:10.1007/978-1-4939-2107-2.

overview of convergence proof

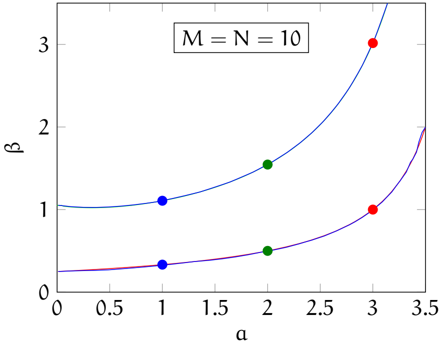
- well-posedness of collocation equation

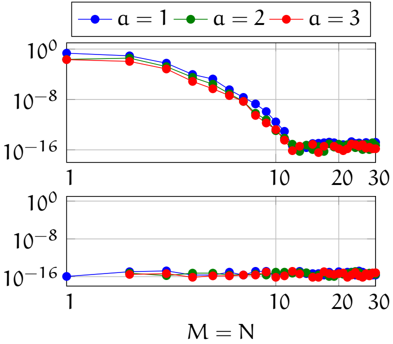
$$(W, Z) = R_N^+ \mathcal{F}_s V((\varphi, \psi), P_N^+(W, Z))$$

- $T_{M,N} \rightsquigarrow \hat{T}_{M,N} := P_M T_{M,N} R_M$ (same eigen-...)
- $\hat{T}_{M,N} = \mathcal{L}_M \hat{T}_N \mathcal{L}_M \rightsquigarrow \hat{T}_N$ ($M \geqslant N$ large $\Rightarrow$ same eigen-...)
- $T, \hat{T}_N \rightsquigarrow T\restriction_{X \times Y_{AC}}, \hat{T}_N\restriction_{X \times Y_{AC}}$ (same eigen-...)
- $\|\hat{T}_N\restriction_{X \times Y_{AC}} - T\restriction_{X \times Y_{AC}}\| \to 0$
- $\Rightarrow$ eigenvalues of $T_{M,N} \to$ eigenvalues of T
  possibly with infinite order
  (smooth functions: error $= O(M^{-k})$ for every k)

logistic *Daphnia* (coupled RE/DDE, equilibria)

$$\begin{cases} b(t) = \beta S(t) \displaystyle\int_a^4 b(t-\sigma)\, d\sigma \\ S'(t) = S(t)\big(1 - S(t)\big) - S(t) \displaystyle\int_a^4 b(t-\sigma)\, d\sigma \end{cases}$$

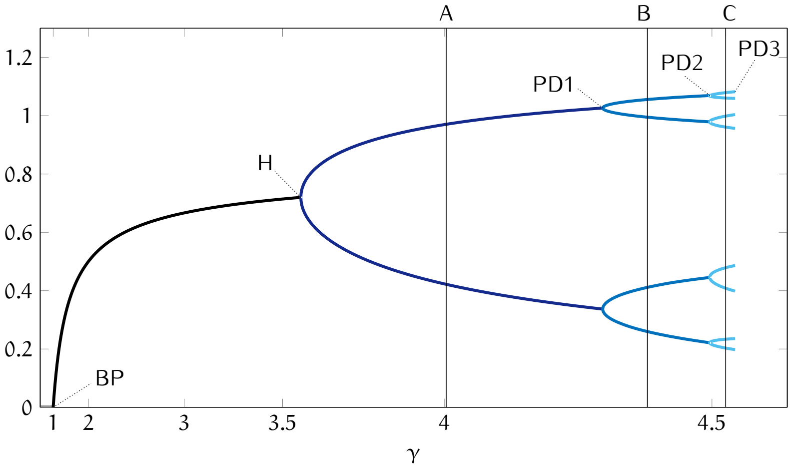




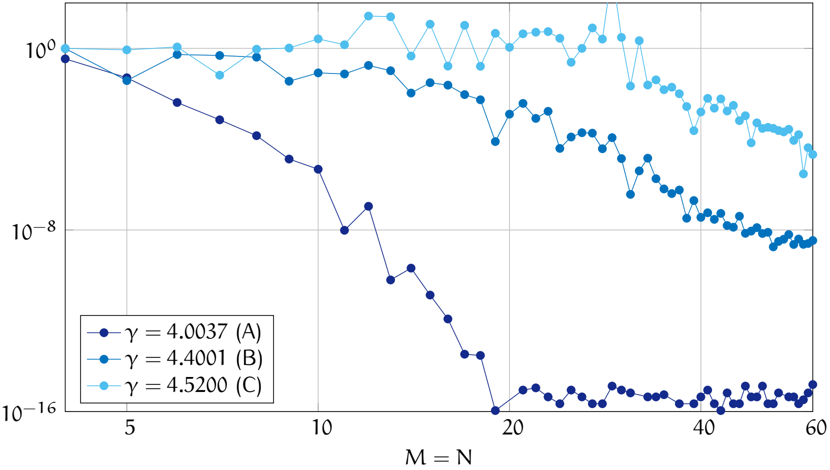
[8] Breda, Diekmann, Maset, and Vermiglio, *A numerical approach for investigating the stability of equilibria for structured population models*, J. Biol. Dyn. 7 (2013), pp. 4–20, DOI:10.1080/17513758.2013.789562.

special RE with quadratic nonlinearity (RE, periodic solutions) [1]

$$x(t) = \frac{\gamma}{2} \int_1^3 x(t-\sigma)\bigl(1 - x(t-\sigma)\bigr)\, d\sigma$$



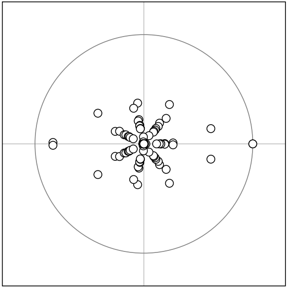
[1] Breda, Diekmann, Liessi, and Scarabel, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ. 65 (2016), pp. 1–24, `DOI:10.14232/ejqtde.2016.1.65`.

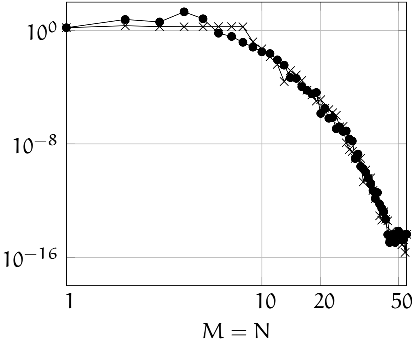


[1] BREDA, DIEKMANN, LIESSI, AND SCARABEL, *Numerical bifurcation analysis of a class of nonlinear renewal equations*, Electron. J. Qual. Theory Differ. Equ. 65 (2016), pp. 1–24, DOI:10.14232/ejqtde.2016.1.65.

$$\begin{cases} x(t) = -\dfrac{1}{2}\left[\displaystyle\int_0^{\frac{7}{2}\pi} x(t-\sigma)\,d\sigma - \int_0^{\frac{\pi}{2}} \ln(y(t-\sigma))\,d\sigma\right] \\ y'(t) = -\ln\left(y\left(t-\dfrac{\pi}{2}\right)\right)y(t) \end{cases}$$

$$(\bar{x}(t), \bar{y}(t)) = (\sin(t), e^{\sin(t)})$$

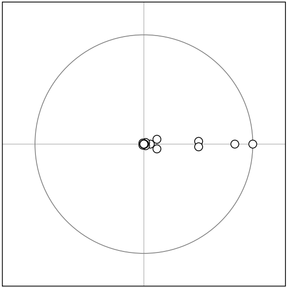


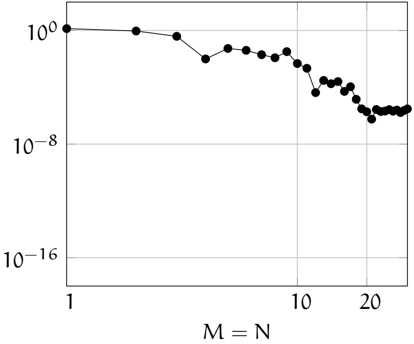


$$A(t) = \beta \int_3^4 A(t-\sigma)\, e^{-A(t-\sigma)}\, d\sigma$$

$$\beta = 7.99896953866859$$

numerically approximated periodic solution





[8] Breda, Diekmann, Maset, and Vermiglio, *A numerical approach for investigating the stability of equilibria for structured population models*, J. Biol. Dyn., 7 (2013), pp. 4–20, DOI:10.1080/17513758.2013.789562.

future perspectives

- apply to realistic models, e.g., *Daphnia* [9]
- extend ideas to Lyapunov exponents (see, e.g., [11])
- extend to neutral equations

Daphnia magna, female [10]

[9] Diekmann, Gyllenberg, Metz, Nakaoka, and De Roos, *Daphnia revisited: local stability and bifurcation theory for physiologically structured population models explained by way of an example*, J. Math. Biol. 61 (2010), pp. 277–318, DOI:10.1007/s00285-009-0299-y.

[10] Watanabe, *Female adult of the water flea Daphnia magna*, PLOS Genetics 7 (2011), issue image, DOI:10.1371/image.pgen.v07.i03.

[11] Breda and Van Vleck, *Approximating Lyapunov exponents and Sacker–Sell spectrum for retarded functional differential equations*, Numer. Math. 126 (2014), pp. 225–257, DOI:10.1007/s00211-013-0565-1.