A DISCRETE COMPETITION-EPIDEMIC MODEL

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Outline

1. Eco-epidemic model
   - Community model.
   - Epidemic model.
   - Complete model.

2. Reduction of two-time scales discrete systems
   - Reduction of two-time scales discrete systems

3. Analysis of the reduced discrete eco-epidemic model
   - Analysis of the reduced discrete eco-epidemic model

Joint work with:

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- **Eva SÁNCHEZ** (Universidad Politécnica de Madrid, SPAIN).
- **Luis SANZ** (Universidad Politécnica de Madrid, SPAIN).

Forming an eco-epidemic model

1. The underlying ecology: Competition Leslie-Gower model.
2. The underlying epidemiology: SIS epidemic model.
3. The underlying interaction of ecology and epidemiology.

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Discrete competition model

Leslie-Gower competition model :

$$N^1(t+1) = \frac{b^1 N^1(t)}{1 + c_{11}N^1(t) + c_{12}N^2(t)}$$

$$N^2(t+1) = \frac{b^2 N^2(t)}{1 + c_{21}N^1(t) + c_{22}N^2(t)}$$

1. $b^i \leq 1$: $N^i(t) \to 0$.
2. $b^1, b^2 > 1$:
    1. $(b^1 - 1)/c_{11} < (b^2 - 1)/c_{21}$: N^1 can be invaded.
    2. $(b^2 - 1)/c_{22} < (b^1 - 1)/c_{12}$: N^2 can be invaded.

Discrete competition model with infected individuals

Distinguishing susceptible and infected individuals in one of the species of the competition model:

$$N_S^1(t+1) = \frac{b_S^1 N_S^1(t)}{1 + c_{SS}N_S^1(t) + c_{SI}N_I^1(t) + c_{S2}N^2(t)}$$

$$N_I^1(t+1) = \frac{b_I^1 N_I^1(t)}{1 + c_{IS}N_S^1(t) + c_{II}N_I^1(t) + c_{I2}N^2(t)}$$

$$N^2(t+1) = \frac{b^2 N^2(t)}{1 + c_{2S}N_S^1(t) + c_{2I}N_I^1(t) + c_{22}N^2(t)}$$

Outline

Discrete SIS epidemic model

The disease process is represented by the map

$$\begin{aligned}\mathbf{F}(N_S, N_I) &= (F_1(N_S, N_I), F_2(N_S, N_I)) \\ &= \left(N_S - \frac{\beta N_S N_I}{N_S + N_I} + \gamma N_I, N_I + \frac{\beta N_S N_I}{N_S + N_I} - \gamma N_I\right)\end{aligned}$$

Conditions for positive solutions: $\gamma \leq 1$ and $\beta < \left(1 + \sqrt{\gamma}\right)^2$.

k outbreaks episodes between two demographic episodes are represented by the k-th iterate:

$$\mathbf{F}^{(k)} = \left(F_S^{(k)}, F_I^{(k)}\right)$$

Long-term behaviour of the SIS epidemic model

The total population size $N = N_S + N_I$ remains constant:

$$F_S(N_S, N_I) + F_I(N_S, N_I) = N_S + N_I$$

Basic reproduction number: $R_0 = \dfrac{\beta}{\gamma}$.

1. $R_0 \leq 1$: Disease-free equilibrium $\mathbf{N}_0^* = (N, 0)$.
2. $R_0 > 1$: Endemic disease.
   If $\gamma < \beta \leq 2 + \gamma$: Stable endemic equilibrium

$$\mathbf{N}_e^* = (N_S^*, N_I^*) = \left(\frac{\gamma}{\beta} N, \left(1 - \frac{\gamma}{\beta} \right) N \right)$$

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Discrete competition-epidemic model with two time scales.

We use $\mathbf{F}^{(k)} = \left(F_S^{(k)}, F_I^{(k)}\right)$ to include k outbreaks episodes between two demographic episodes

$$N_S^1(t) = F_S^{(k)}\big(N_S^1(t), N_I^1(t)\big) \text{ and } N_I^1(t) = F_I^{(k)}\big(N_S^1(t), N_I^1(t)\big)$$

$$N_S^1(t+1) = \frac{b_S^1 N_S^1(t)}{1 + c_{SS} N_S^1(t) + c_{SI} N_I^1(t) + c_{S2} N^2(t)}$$

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$$N^2(t+1) = \frac{b^2 N^2(t)}{1 + c_{2S} N_S^1(t) + c_{2I} N_I^1(t) + c_{22} N^2(t)}$$

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Outline

Slow-fast discrete models

The dynamics of the population consists of two processes defined by two maps

$$F, S : \Omega_N \longrightarrow \Omega_N \quad ; \quad F, S \in C^1(\Omega_N), \quad \Omega_N \subset \mathbb{R}^N$$

Two time scales: Fast and slow processes

$$X_k(t+1) = S(F^{(k)}(X_k(t))).$$

$F^{(k)}$ is k-th iterate of F.

- R. Bravo de la Parra, M. Marvá, E. Sánchez, L. Sanz
  Reduction of Discrete Dynamical Systems with Applications to Dynamics Population Models
  Mathematical Modelling of Natural Phenomena, **8**(6):107-129, 2013.

Slow-fast discrete models : eco-epidemic model

Let S be the map associated to the system (slow part)

$$
\begin{aligned}
N_S^1(t+1) &= b_S^1 N_S^1(t) / \left(1 + c_{SS} N_S^1(t) + c_{SI} N_I^1(t) + c_{S2} N^2(t)\right) \\
N_I^1(t+1) &= b_I^1 N_I^1(t) / \left(1 + c_{IS} N_S^1(t) + c_{II} N_I^1(t) + c_{I2} N^2(t)\right) \\
N^2(t+1) &= b^2 N^2(t) / \left(1 + c_{2S} N_S^1(t) + c_{2I} N_I^1(t) + c_{22} N^2(t)\right)
\end{aligned}
$$

and F the map (fast part)

$$F(N_S^1, N_I^1, N^2) = \left(F_1(N_S^1, N_I^1), F_2(N_S^1, N_I^1), N^2\right)$$

then

$$F^{(k)}(N_S^1, N_I^1, N^2) = \left(F_1^{(k)}(N_S^1, N_I^1), F_2^{(k)}(N_S^1, N_I^1), N^2\right)$$

and the complete system has the form

$$X_k(t+1) = S(F^{(k)}(X_k(t)))$$

with $X = (N_S^1, N_I^1, N^2)$.

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Slow-fast discrete models: Reduced system

Two hypotheses on fast dynamics:

1. $\{F^{(k)}\}_{k\in\mathbb{N}}$, converges to a map $\bar{F} : \Omega_N \to \Omega_N$, $\bar{F} \in C^1(\Omega_N)$.

   Approximation of the system: $X(t+1) = S \circ \bar{F}(X(t))$

2. There exist a non-empty open subset $\Omega_q \subset \mathbb{R}^q$ with $q < N$ and two maps $G : \Omega_N \longrightarrow \Omega_q$ and $E : \Omega_q \longrightarrow \Omega_N$, $G \in C^1(\Omega_N)$ and $E \in C^1(\Omega_q)$, such that $\bar{F} = E \circ G$.

   $X(t+1) = S \circ E \circ G(X(t))$

Global (slow) **variables** $Y := G(X) \in \mathbb{R}^q$.

The **reduced or aggregated system**

$$Y(t+1) = (G \circ S \circ E)(Y(t)) := \bar{S}(Y(t)).$$

Slow-fast discrete model: Reduced eco-epidemic model

$$\bar{F}\left(N_S^1, N_I^1, N^2\right) = \left(\nu(N_S^1 + N_I^1), (1-\nu)(N_S^1 + N_I^1), N^2\right) = E \circ G\left(N_S^1, N_I^1, N^2\right)$$

$$G\left(N_S^1, N_I^1, N^2\right) = \left(N_S^1 + N_I^1, N^2\right) = \left(N^1, N^2\right) = Y \qquad \text{(Global variables)}$$

$$E\left(N^1, N^2\right) = \left(N^1\nu, N^1(1-\nu), N^2\right)$$

where $\nu = 1$ if $R_0 \leq 1$ and $\nu = 1/R_0$ if $R_0 > 1$,

The reduced system $Y(t+1) = (G \circ S \circ E)(Y(t))$:

Generalized Leslie-Gower competition model

$$N^1(t+1) = \frac{\nu b_S^1 N^1(t)}{1 + \left(\nu c_{SS} + (1-\nu)c_{SI}\right) N^1(t) + c_{S2}N^2(t)} + \frac{(1-\nu) b_I^1 N^1(t)}{1 + \left(\nu c_{IS} + (1-\nu)c_{II}\right) N^1(t) + c_{I2}N^2(t)}$$

$$N^2(t+1) = \frac{b^2 N^2(t)}{1 + \left(\nu c_{2S} + (1-\nu)c_{2I}\right) N^1(t) + c_{22}N^2(t)}$$

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   $X(t+1) = S \circ E \circ G(X(t))$

Global (slow) **variables** $Y := G(X) \in \mathbb{R}^q$.

The **reduced or aggregated system**

$$Y(t+1) = (G \circ S \circ E)(Y(t)) := \bar{S}(Y(t)).$$

Reduction of nonlinear discrete systems

Hypotheses of useful reduction:

$$\lim_{k\to\infty} F^{(k)} = \bar{F} \text{ and } \lim_{k\to\infty} DF^{(k)} = D\bar{F}$$

uniformly on any compact set $K \subset \Omega_N$.

$Y^* \in \mathbb{R}^q$ hyperbolic equilibrium point of the reduced system $Y_{n+1} = \bar{S}(Y_n)$.

Then for k large enough system $X_k(t+1) = S(F^{(k)}(X_k(t)))$ possesses a hyperbolic equilibrium point X_k^* satisfying $\lim_{k\to\infty} X_k^* = X^* = S(E(Y^*))$.

1. If Y^* is asymptotically stable then X_k^* is a.s.. If $X_0 \in \mathbb{R}^N$ is such that $\lim_{n\to\infty} \bar{S}^n(G(X_0)) = Y^*$, then $\lim_{n\to\infty} \left(S(F^{(k)})\right)^n (X_0) = X_k^*$.
2. If Y^* is unstable then X_k^* is unstable, for each $k \geq k_0$.

An analogous result can be stated for periodic solutions

- L. Sanz, R. Bravo de la Parra, E. Sánchez
  Approximate Reduction of Non-Linear Discrete Models with Two Time Scales
  Journal of Difference Equations and Applications, **14**(6):607-627, 2008.

Reduction of an eco-epidemic model

The map F associated to the disease process verifies de reduction hypotheses

$$\lim_{k\to\infty} F^{(k)}(N_S^1, N_I^1, N^2) = \bar{F}(N_S^1, N_I^1, N^2) = \left(\nu(N_S^1 + N_I^1), (1-\nu)(N_S^1 + N_I^1), N^2\right)$$

where $\nu = 1$ if $R_0 = \dfrac{\beta}{\gamma} \leq 1$ and $\nu = \dfrac{\gamma}{\beta} = \dfrac{1}{R_0}$ if $R_0 > 1$.

$$\lim_{k\to\infty} DF^{(k)}\left(N_S^1, N_I^1, N^2\right) = D\bar{F}\left(N_S^1, N_I^1, N^2\right) = \begin{pmatrix} \nu & \nu & 0 \\ 1-\nu & 1-\nu & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

both limits are uniform on compact sets. The uniform convergence can be proved because we can write F in terms of the scalar function

$$\phi(x) = x\left(1 + \beta(1-x) - \gamma\right)$$

(updating the fraction of infected individuals)

Outline

Reduced Discrete SIS-competition model

$$N^1(t+1) = \frac{\nu b_S^1 N^1(t)}{1 + (\nu c_{SS} + (1-\nu)c_{SI})\,N^1(t) + c_{S2}N^2(t)} + \frac{(1-\nu)b_I^1 N^1(t)}{1 + (\nu c_{IS} + (1-\nu)c_{II})\,N^1(t) + c_{I2}N^2(t)}$$

$$N^2(t+1) = \frac{b^2 N^2(t)}{1 + (\nu c_{2S} + (1-\nu)c_{2I})\,N^1(t) + c_{22}N^2(t)}$$

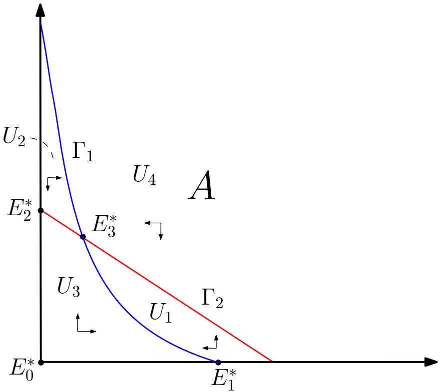
1. All solutions in $\mathbf{R}_+^2$ are forward bounded.
2. The system is strongly competitive in $\mathbf{R}_+^2$.
3. All solutions in $\mathbf{R}_+^2$ are eventually componentwise monotone, and tend to an equilibrium point.

If $\nu b_S^1 + (1-\nu)b_I^1 \leq 1$ then $N^1(t) \to 0$ and if $b^2 \leq 1$ then $N^2(t) \to 0$.

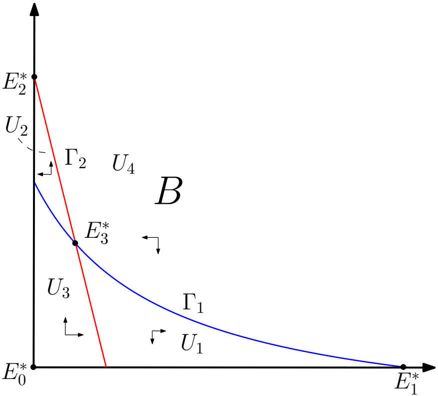
- R. Bravo de la Parra, M. Marvá, E. Sánchez, L. Sanz
  Discrete Models of Disease and Competition
  Discrete Dynamics in Nature and Society, Volume 2017, Article ID 5310837, 13 pages, https://doi.org/10.1155/2017/5310837.

Reduced Discrete SIS-competition model

$\nu b_S^1 + (1-\nu) b_I^1 > 1$ and $b^2 > 1$, i.e., E_1^* and E_2^* exist:

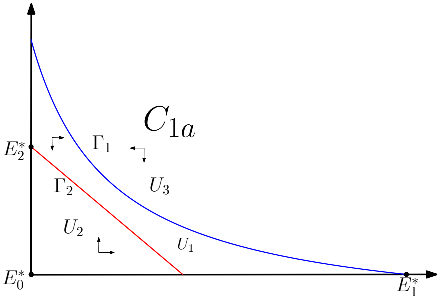


(a) **Case A**. Coexistence.

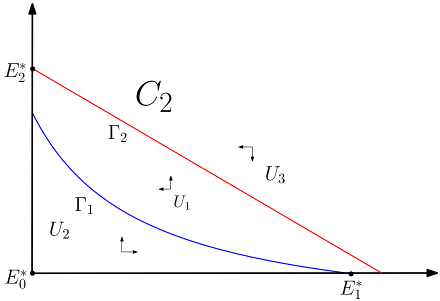


(b) **Case B**. N^1 or N^2 exclusion depending on initial conditions.

Reduced Discrete SIS-competition model



(a) **Case $\mathbf{C_{1a}}$**. N^2 exclusion.



(b) **Case $\mathbf{C_2}$**. N^1 exclusion.

Reduced Discrete SIS-competition model

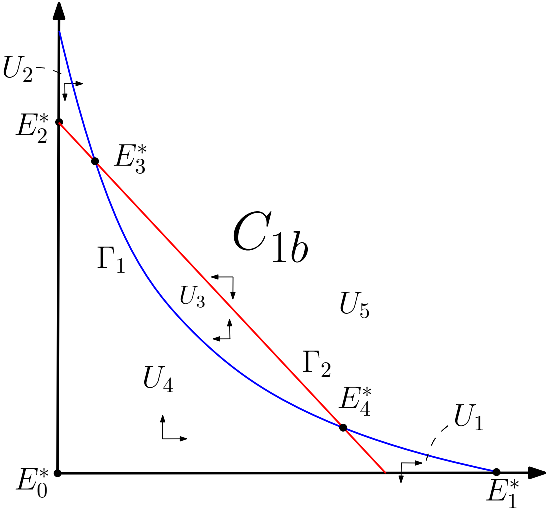


Figure: **Case $\mathbf{C_{1b}}$**. N^2 exclusion or coexistence depending on initial conditions.

Discrete competition epidemic model

Disease modified growth capabilities

$$c_{11} := c_{SS} = c_{SI} = c_{IS} = c_{II}, \quad c_{12} := c_{S2} = c_{I2} \quad \text{and} \quad c_{21} = c_{2S} = c_{2I}.$$

Case A (coexistence).

$$b_S^1 = b^1 \text{ and } b_I^1 = \alpha b^1$$

$\alpha > 0$ measures the effect of the disease.
Ratio of the species 1 equilibrium with and without disease

$$\frac{c_{22}\left(b^1[\nu+(1-\nu)\alpha]-1\right)-c_{12}\left(b^2-1\right)}{c_{22}(b^1-1)-c_{12}(b^2-1)}$$

For species 2

$$\frac{c_{11}\left(b^2-1\right)-c_{21}\left(b^1[\nu+(1-\nu)\alpha]-1\right)}{c_{11}(b^2-1)-c_{21}(b^1-1)}$$

If $\alpha < 1$ (resp., $\alpha > 1$) the long term population size of species 1 is reduced (resp., increased) and the opposite happens to species 2.

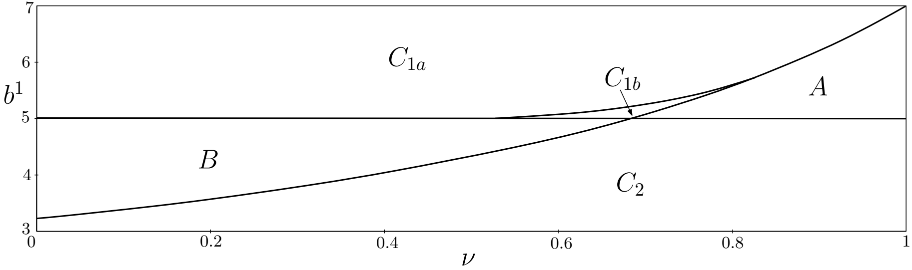
Discrete competition epidemic model

Disease modified competitive abilities

$$b^1 := b^1_S = b^1_I > 1,$$

$$c_{SS} = 3 > c_{SI} = 2.8 \text{ and } c_{2S} = 2 > c_{2I} = 1.8,$$

$$b^2 = 5,\ c_{S2} = c_{IS} = c_{II} = c_{I2} = c_{22} = 1, \text{ and } \nu \in (0, 1].$$



Increasing $R_0 = 1/\nu$ improves the species 1 competition outcome.

THANK YOU!

Rafa Bravo