Optimal Control of Invasive Species

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Work in collaboration with
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Outline

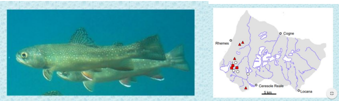
1. The ecological problem of invasive species in protected areas
   - The spread of invasive species
   - The control actions
2. An optimal control model for invasive species
   - Model formulation
   - Existence - necessary conditions - uniqueness
   - Analysis of the state-control optimality system
   - The free terminal time problem
3. Further developments
   - A non-autonomous optimal control problem
   - Spatio-temporal control with budget constraint
4. Conclusion

The spread of invasive species

Invasive species are alien species (e.g. transported for leisure hunting/fishing purposes) that supersede native species causing ecological damage to the recipient ecosystem. Examples:

Orange hawkweed and feral cat in Victorian Alpine National Park (Australia)

Tree of heaven and wild-boar in Alta Murgia National Park (Italy)



Brook trout in Gran Paradiso National Park (Italy)

The control actions

- *Piano di gestione triennale del cinghiale nel Parco Nazionale dell'Alta Murgia- Delibera n. 21/2012 del 18/12/2012.*
- LIFE+ Alta Murgia - Control and eradication of the invasive and exotic plant species *Ailanthus altissima* in the Alta Murgia National Park.
- LIFE+ BIOAQUAE - Biodiversity Improvement of Aquatic Alpine Ecosystems: eradication of non-native fish species from some high altitude alpine lakes.

We search for the best control effort and allocation strategy for eradicating/reducing the abundance of the invasive species in the support of an enhanced decision making.

Model formulation

We focus on the temporal aspects of management with the assumption that the abundance of the species has a much greater influence on the dynamics than its spatial distribution.
The dynamics of the invasive population is described by the ODE

$$\dot{u}(t) = u\,f(u) - u\,(\mu\,E)^q\,, \quad t \in [0, T]$$

where

- $u\,f(u)$ represents the density-dependent population growth.
- $u\,(\mu\,E)^q$ represents the effects of the control actions E on the population dynamics:
  - $\mu > 0$ is a scaling parameter which accounts for the control effectiveness;
  - $q \in \mathbb{Q} \cap \left[\frac{1}{2}, 1\right)$ is a diminishing returns parameter.

Baker, C.M. and M. Bode, *Placing invasive species management in a spatiotemporal context*, Ecological Applications, **26**, 2016, 712–725.

The study cases

$$r\,u\left(1-\frac{u}{k}\right)$$

feral cats in Australia

logistic growth

$$uf(u) = \begin{matrix} \nearrow \\ \\ \searrow \end{matrix}$$

$$r\,u\left(\frac{u}{k_0}-1\right)\left(1-\frac{u}{k}\right)$$

wild-boars in Italy[3]

growth with Allee effect

r the intrinsic growth rate, k the carrying capacity, k_0 the Allee threshold

[3] Lacitignola, D., Diele, F., Marangi, C., Dynamical scenarios from a two-patch predator prey system with human control - Implications for the conservation of the wolf in the Alta Murgia National Park, Ecol. Model. **316**, 2015, 28–40.

The optimal control problem

The control set is

$$U_b = \left\{ E \in L^1(0,T) : \, 0 \leq E \leq b \right\}, \quad b > 0$$

We search for the optimal control function $E^* \in U_b$ which realizes

$$\min_{E \in U_b} \int_0^T \mu \, E(t) \, dt$$

subject to the state equation

$$\dot{u} = u \, f(u) - u \, (\mu \, E)^q, \quad t \in [0, T]$$

with fixed initial and final conditions

$$u(0) = u_0, \quad u(T) = u_T, \quad 0 < u_T < u_0 \leq k$$

Theorem

For T sufficiently large, the minimization problem has an optimal solution.[a]

[a] Baker C.M., F. Diele, D. Lacitignola, C. Marangi, A. Martiradonna. Optimal Control of Invasive Species through a Dynamical Systems Approach, Discrete & Continuous Dynamical Systems - B, 2017, under revision.

Proof.[4]

- The right hand side of the state equation, $\tilde{f}(u, E) = r\,u\,(1 - \dfrac{u}{k}) - u\,\mu^q\,E^q$, is continuous in (u, E) and continuously differentiable w.r.t. u. Moreover $|\widetilde{f}(u, E)| \leq C(1 + |u|)$.
- The set of the admissible controls steering the state variable from u_0 to u_T, at the time T, is nonempty.
- The sets $\{(y, y_0) \in \mathbb{R}^2 : y = \tilde{f}(u, E),\ y_0 \geq \mu\,E,\ \textit{for some } E \in U_b\}$ are convex, for all u.

[4] Bressan, A., Piccoli B., Introduction to the Mathematical Theory of Control, AIMS Series on Applied Mathematics, 2, Springfield, 2007.

First-order necessary conditions for optimality

Let (E^*, u^*) be an optimal solution. There exists a piecewise differentiable function $\lambda > 0$ such that

$$H(u^*(t), E^*(t), \lambda(t)) \leq H(u^*(t), E(t), \lambda(t)), \quad \forall E \in U_b,\ t \in [0, T],$$

where

$$H(u, E, \lambda) = \mu\, E \,+\, \lambda\, u\, f(u) - \lambda\, u\, \mu^q\, E^q$$

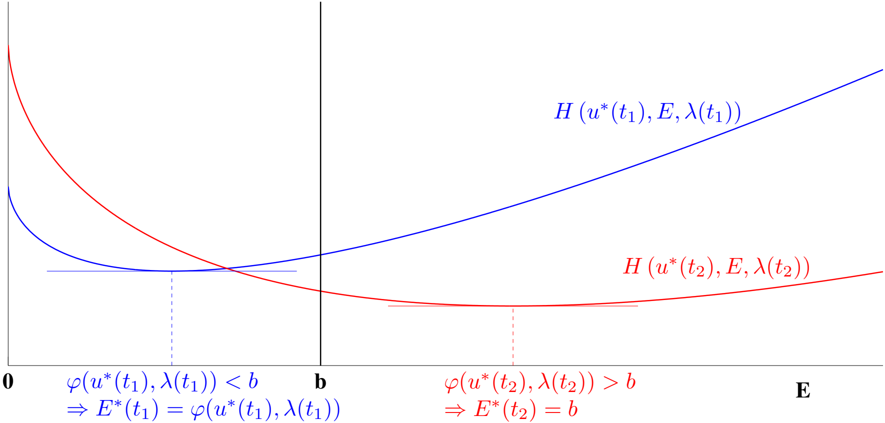
is the Hamiltonian function. Moreover λ obeys the adjoint equation

$$\dot{\lambda} = -\lambda\, f(u^*) - \lambda\, u^*\, f'(u^*) + \,\lambda\, \mu^q\, E^{*q}$$

and, since $\lim_{E \to 0^+} \frac{\partial H}{\partial E}\,(u^*, E, \lambda) \,= -\infty$,

$$\begin{cases} 0 < E^* \leq b & \text{if} \quad \frac{\partial H}{\partial E}\,(u^*, E^*, \lambda) = \mu - q\,\lambda\; u^*\,\mu^q\, E^{*\,q-1} = 0 \\[2ex] E^* = b & \text{if} \quad \frac{\partial H}{\partial E}\,(u^*, b, \lambda) = \mu - q\,\lambda\; u^*\,\mu^q\, b^{q-1} <\, 0 \end{cases}$$

$$E^*(t) = \min\{\varphi(u^*(t), \lambda(t)), b\}, \text{ where } \varphi(u,\lambda) = \frac{1}{\mu}\,(q\,u\,\lambda)^{\frac{1}{1-q}}$$



CONVEXITY:

$$\frac{\partial^2 H}{\partial E^2}\,(u, E, \lambda) = q\,(1-q)\,\lambda\;u\,\mu^q\,E^{q-2} > 0, \quad u,\ \lambda,\ E > 0.$$

Theorem (Baker, Diele, Lacitignola, Marangi, Martiradonna, 2017)

For T sufficiently small, bounded solutions of the state-adjoint system

$$\dot{u} = u\,f(u) - u\,\mu^q\,{E^*_{u,\lambda}}^q, \quad u(0) = u_0, \quad u(T) = u_T,$$
$$\dot{\lambda} = -\lambda\,f(u) - \lambda\,u\,f'(u) + \lambda\,\mu^q\,{E^*_{u,\lambda}}^q,$$

with $E^*_{u,\lambda}(t) = \min\left\{\frac{1}{\mu}\,(q\,u(t)\,\lambda(t))^{\frac{1}{1-q}}\,, b\right\}$, *are unique.*

Proof. $(u,\,\lambda) = (e^{\epsilon\,t}\,p, e^{-\epsilon\,t}\,w)$, $(\bar{u},\,\bar{\lambda}) = (e^{\epsilon\,t}\,\bar{p}, e^{-\epsilon\,t}\,\bar{w})$ two solutions, $\epsilon > 0$.
Boundedness of the solutions and the assumption $\frac{1}{2} \leq q < 1$, give

$$|{E^*_{u,\lambda}}^q - {E^*_{\bar{u},\bar{\lambda}}}^q| \leq C\,|u\,\lambda - \bar{u}\,\bar{\lambda}| = C|p\,w - \bar{p}\,\bar{w}|, \quad C > 0.$$

After some calculations:

$$(\epsilon - \tilde{C}_1 - \tilde{C}_2\,e^{2\,\epsilon\,T}) \int_0^T \left[(p-\bar{p})^2 - (w-\bar{w})^2\right]\,dt \leq 0, \quad \tilde{C}_1,\,\tilde{C}_2 > 0.$$

For $\epsilon > \tilde{C}_1 + \tilde{C}_2$ and $T < \dfrac{1}{2\,\epsilon}\log\dfrac{\epsilon - \tilde{C}_1}{\tilde{C}_2}$, uniqueness holds.

The state-control optimality system

State-adjoint system
$$\boxed{\begin{aligned}\dot u &= u\,f(u) - u\,(q\,u\,\lambda)^{\frac{q}{1-q}}\\ \dot\lambda &= -\lambda\,f(u) - \lambda\,u\,f'(u) + \lambda\,(q\,u\,\lambda)^{\frac{q}{1-q}}\end{aligned}}$$

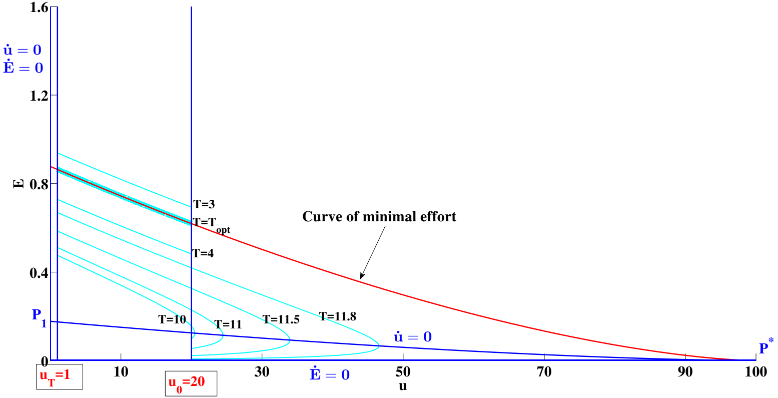
$$\downarrow$$

State-control system
$$\boxed{\begin{aligned}\dot u &= u\,f(u) - u\,\mu^q\,E^q\\ \dot E &= \tfrac{1}{q-1}\,u\,f'(u)\,E\end{aligned}}$$

Invariant for the state-control system: $I(u,E) = (q-1)\,E + \dfrac{f(u)}{\mu^q}\,E^{1-q}$

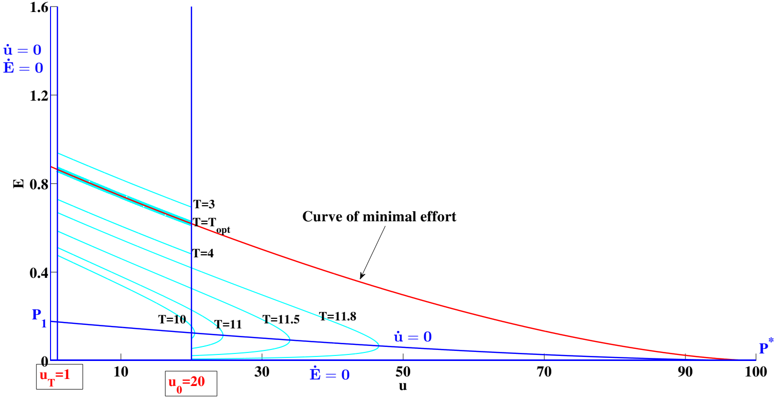
$$I(u,E) = I(k,0) = 0 \Rightarrow \begin{cases} E = 0 & \text{the stable manifold of the saddle } P^* = (k,0) \\ E = \dfrac{1}{\mu}\left(\dfrac{f(u)}{1-q}\right)^{\frac{1}{q}} & \text{the unstable manifold of } P^* \end{cases}$$

The state-control optimality system. Logistic growth.



$r = 0.55, \ k = 100, \ \mu = 2.21$

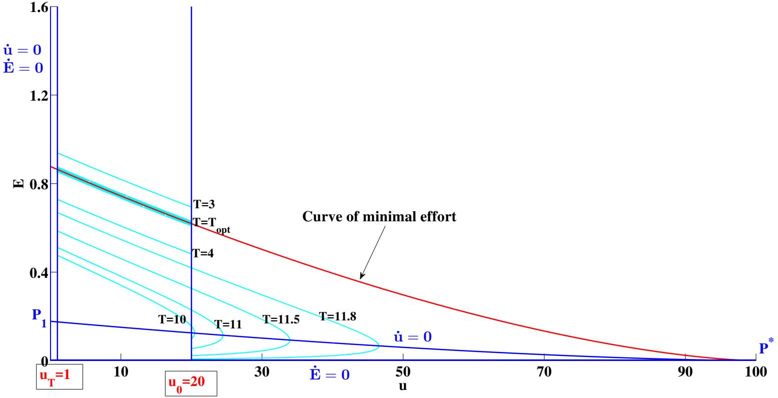
The state-control optimality system. Logistic growth.



$r = 0.55, \ k = 100, \ \mu = 2.21$

Which is the optimal time horizon?

The state-control optimality system. Logistic growth.



$$r = 0.55, \;\; k = 100, \;\; \mu = 2.21$$

Which is the optimal time horizon?

$$I(u^*(T^*), E^*(T^*)) = 0 \Rightarrow E^* = \frac{1}{\mu}\left(\frac{r}{1-q}\left(1 - \frac{u^*}{k}\right)\right)^{\frac{1}{q}} \text{ curve of minimal effort}$$

Theorem (Free terminal time, logistic growth)

Let $\mu > 0$ *be a fixed constant,* $\bar{T} > 0$ *arbitrarily large, and consider the control set* $U_b = \{E \in L^1(0,T) : 0 \leq E \leq b,\ 0 \leq T \leq \bar{T}\}$, *with* $b > \frac{1}{\mu}\left(\frac{r}{1-q}\right)^{1/q}$ *and* $q \in \mathbb{Q}$ *with* $\frac{1}{2} \leq q < 1$. *If* $0 < u_T < u_0 < k$, *the minimization problem*

$$\min_{(E,T)\in U_b\times[0,\bar{T}]} \int_0^T \mu E(t)\,dt$$

subject to $\quad \dot{u} = r\,u(1-\frac{u}{k}) - u\,\mu^q\,E^q,\ 0 \leq t \leq T, \quad u(0) = u_0,\ u(T) = u_T,$

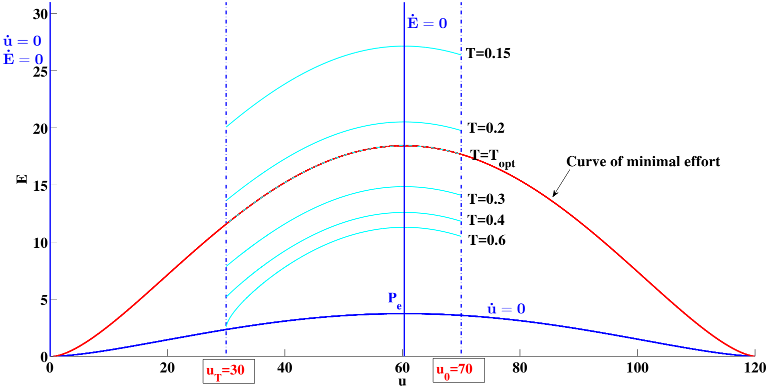
has the optimal solution (E^*, T^*), *where* $T^* = \dfrac{1-q}{r\,q}\log\dfrac{u_0(k-u_T)}{u_T(k-u_0)}$ *and*

$$E^*(t) = \frac{1}{\mu}e^{\frac{r}{1-q}(t-T^*)}\left(\frac{r\,(k-u_T)}{(1-q)\,\left[u_T-(u_T-k)\,e^{\frac{rq}{1-q}(t-T^*)}\right]}\right)^{\frac{1}{q}}, \quad t \in [0,T^*].$$

Moreover, the optimal density solution is provided by

$$u^*(t) = \frac{u_T\,k}{u_T-(u_T-k)\,e^{rq(t-T^*)/(1-q)}}, \quad t \in [0,T^*].$$

The state-control optimality system. Growth with Allee.



$$r = 0.0484,\ k = 120,\ k_0 = 0.6182,\ \mu = 2.21$$

The optimal time horizon:

$$I(u^*(T^*), E^*(T^*)) = 0 \Rightarrow E^* = \frac{1}{\mu}\left(\frac{r}{1-q}\left(\frac{u^*}{k_0}-1\right)\left(1-\frac{u^*}{k}\right)\right)^{\frac{1}{q}} \text{ curve of minimal effort}$$

Theorem (Free terminal time, growth with Allee)

Let $\mu > 0$ *be a fixed constant,* $\bar{T} > 0$ *arbitrarily large, and consider the control set* $U_{0,b} = \{E \in L^1(0,T) : 0 \le E \le b,\ 0 \le T \le \bar{T}\}$, *with* $b \ge \frac{1}{\mu}\left(\frac{r\,(k-k_0)^2}{2\,(1-q)\,k_0\,k}\right)^{1/q}$, *and* $q \in \mathbb{Q}$, *with* $\frac{1}{2} \le q < 1$. *If* $k_0 < u_T < u_0 < k$, *the minimization problem*

$$\min_{(E,T)\in U_b\times[0,\bar{T}]} \int_0^T \mu\, E(t)\, dt$$

subject to $\dot{u} = r\,u\left(\frac{u}{k_0}-1\right)\left(1-\frac{u}{k}\right) - u\,\mu^q\,E^q \quad 0 \le t \le T,\ u(0) = u_0,\ u(T) = u_T$, *has the optimal solution* (E^*, T^*), *where* $T^* = \frac{1}{A_1}\log\left[\left(\frac{u_T}{u_0}\right)^B\left(\frac{u_0-k_0}{u_T-k_0}\right)^C\left(\frac{k-u_T}{k-u_0}\right)^D\right]$, *and*

$$E^*(t) = \frac{1}{\mu}\left[\frac{r}{1-q}\left(\frac{u^*(t)}{k_0}-1\right)\left(1-\frac{u^*(t)}{k}\right)\right]^{1/q}, \quad t \in [0, T^*].$$

Moreover, the optimal solution $u^*(t)$ *satisfies*

$$\frac{(u-k_0)^C}{u^B\,(k-u)^D} = \frac{(u_{T^*}-k_0)^C}{u_T^B\,(k-u_{T^*})^D}\,e^{A_1(T^*-t)}, \quad t \in [0, T^*].$$

with $A_1 = \frac{r\,q}{1-q}$, $B = \frac{1}{k\,k_0}$, $C = \frac{1}{k_0(k-k_0)}$, $D = \frac{1}{k(k-k_0)}$.

A non-autonomous optimal control problem

Introduce a discount factor $\delta > 0$ in the objective functional.

$$\mathcal{J}(E^*, T^*) = \min_{E \in U_b,\, T \in [0, \bar{T}]} \int_0^T e^{-\delta\, t}\, E(t)\, dt$$

$$\dot{u} = r\, u \left(1 - \frac{u}{k}\right) - u\, (\mu\, E)^q, \quad u(0) = u_0, \quad u(T) = u_T < u_0 \leq k$$

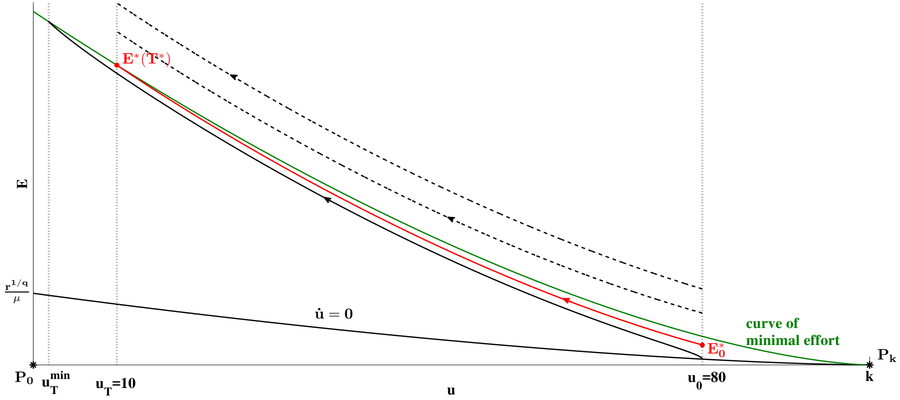
Results:

- $\mathcal{I}(t, u, E) = \dfrac{e^{-\delta\, t}}{q} \left[(q-1)\, E + \dfrac{r}{\mu^q} \left(1 - \dfrac{u}{k}\right) E^{1-q}\right] + \delta \displaystyle\int_0^t e^{-\delta\, s}\, E(s)\, ds,$ is an invariant for the state-control system;
- $(q-1) E^*(T^*) + \dfrac{r}{\mu^q} \left(1 - \dfrac{u_T}{k}\right) E^*(T^*)^{1-q} = 0.$

Then $\quad \mathcal{J}(E^*, T^*) = \dfrac{1}{\delta\, q} \left[q\, E_0^* - \left({E_0^*}^q - \dfrac{r}{\mu^q} \left(1 - \dfrac{u_0}{k}\right)\right) {E_0^*}^{1-q}\right]$

Martiradonna A., Diele F., Marangi C., Analysis of state-control optimality system for invasive species management, Springer Proceedings in Mathematics & Statistics, ISAAC 2017, submitted.

A non-autonomous optimal control problem



Spatio-temporal control with budget constraint

Fix the project length T, introduce the spatial domain $\Omega \in \mathbf{R}^2$ and a budget constraint $B > 0$ in the objective function. Find

$$E^* \in \mathcal{U} = \{E \in L^\infty(\Omega \times [0,T]) \ : \ 0 \leq E(\mathbf{x},t) \leq B \text{ for all } (\mathbf{x},t) \in \Omega \times [0,T]\}$$

which realizes

$$\min_{E \in \mathcal{U}} \left[\int_\Omega e^{-\delta T} \nu(\mathbf{x})\, u \, d\mathbf{x} + \int_{\Omega \times [0,T]} e^{-\delta t} \left(\omega(\mathbf{x},t)\, u + E^q + c \left(\frac{E}{B} \right)^{2q-1} \right) d\mathbf{x}\, dt \right],$$

subject to the dynamics

$$\frac{\partial u}{\partial t} - D\,\Delta u = r\,u \left(\rho(\mathbf{x}) - \frac{u}{k} \right) - \frac{\mu\,u\,E}{1 + h\,\mu\,u}, \quad (\mathbf{x},t) \in \Omega \times [0,T]$$
$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ \mathbf{x} \in \Omega, \quad \nabla u \cdot \mathbf{n} = 0,\ \text{on } \partial\Omega \times [0,T].$$

Baker C.M., F. Diele, C. Marangi, A. Martiradonna, S. Ragni. Optimal control governed by a diffusion PDE with Holling type II reaction term and budget constraint. Natural Resource Modeling, 2018, submitted.

Optimality system:

$$\frac{\partial u}{\partial t} - D\,\Delta u = r\,u\left(\rho(\mathbf{x}) - \frac{u}{k}\right) - \frac{\mu\,u\,E^*}{1+h\,\mu\,u},$$
$$u(\mathbf{x},0) = u_0(\mathbf{x}),\ \mathbf{x}\in\Omega, \quad \nabla u\cdot\mathbf{n} = 0,\ \text{on } \partial\Omega\times[0,T].$$

$$\frac{\partial \lambda}{\partial t} + D\,\Delta\lambda = \delta\,\lambda - r\,\rho(\mathbf{x})\,\lambda + \frac{2\,r}{k}\,u\,\lambda + \frac{\mu\,E^*\,\lambda}{(1+h\,\mu\,u)^2} - \omega,$$
$$\lambda(\mathbf{x},T) = \nu(\mathbf{x}),\ \mathbf{x}\in\Omega, \quad \nabla\lambda\cdot\mathbf{n} = 0,\ \text{on } \partial\Omega\times[0,T].$$

$$E^*(\mathbf{x},t) = \min\{\varphi_\alpha(n^*(\mathbf{x},t),\lambda(\mathbf{x},t)), B\}$$

where

$$\varphi_\alpha(s,z) = \begin{cases} \left[\frac{q}{2\alpha}\left(\sqrt{1+\frac{4\,\alpha\,\mu\,s\,z}{q^2\,(1+h\,\mu\,s)}} - 1\right)\right]^{\frac{1}{q-1}}, & \text{if } \alpha > 0, \\ \left(\frac{\mu\,s\,z}{q\,(1+h\,\mu\,s)}\right)^{\frac{1}{q-1}}, & \text{if } \alpha = 0, \end{cases} \tag{1}$$

for each $s, z \geq 0$ and $\alpha = c\,(2q-1)/B^{2q-1}$.

The approximation uses a semi-discretization in the space variable performed by FE method; a splitting and composing procedure in forward-backward form for the diffusive and the reaction term. Symplectic-exponential Lawson procedure for integrating the reaction term.

Conclusion

Taking into account the spatio-temporal feature of the invasive species over large and irregular environments is a challenging task in optimal control of invasive species.

Ongoing applications:

Ailanthus altissima (tree of heaven)
in Alta Murgia National Park, Italy

Salvelinus fontinalis (brook trout)
in Gran Paradiso National Park, Italy

The model may be further enriched by taking into account the age. The fish age-structured model presented in (Marinoschi G., Martiradonna A., Fish populations dynamics with nonlinear stock-recruitment renewal conditions. Applied Mathematics and Computation, 2016, 277: 101-110.) might be the proper choice e.g. to treat the case of the brook trout.

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Thank you for the attention